Aquifer Characteristics for Predicting Groundwater Table Using van Deemter's Analysis

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ABSTRACT

Efficient water use on agricultural land requires detailed knowledge of the water needs of the cropping system , the hydrology of the land area concerned, and the effectiveness of water conveyance in the irrigation or drainage system used. Much research has been conducted over the years in these areas of concern. Less is known how variations in the water level of the conveyance system affects the groundwater levels and to have a better basis for determining the required frequency, size, and density of the water conveyance system in order to ensure an adequate water supply for meeting the crop needs everywhere in the field or for the efficient removal of excess groundwater through drainage.

FLOW REGION AND SOLUTION

The historic analysis of van Deemter (1950) allows a precise evaluation of groundwater levels of drainage and sub-irrigation levels in aquifers of land areas between equidistantly placed drains and/or ditches. His analysis was performed for the case of steady state flow with constant and uniform rainfall and/or evapo-transpiration rates using a solution of the Laplace equation for a two-dimensional flow field with an infinitely deep homogeneous and isotropic aquifer intersected to a finite depth with a system of equidistantly placed drains or ditches. Contrary to traditional approximate methods for this fairly simple flow field problem , van Deemter (1950) used the lesser known but potentially the more powerful conformal approach in which the geometric flow field, represented by the complex space variable $z= x + i\cdot y$ and the complex flow potential field ω for this area, represented by the functional relationship $\omega = \phi + i\cdot\psi$, are projected on a complex plane, say t, so that the corresponding vertices of the geometric and potential planes coincide. ϕ and ψ are the pressure and stream flow component potentials, respectively. In van Deemster 's approach, the space variable z and the potential variable ω for the vertices of the flow field are transformed, directly or indirectly, through appropriate relationships onto the upper half of the complex t-plane and co-incide and are located on the real axis of the t-plane. For most flow fields this is a daunting task. One of the earliest published works using this technique on agricultural land was by Muskat and Wyckoff (1937). The technique is more commonly used in groundwater problems involving streams with drop structures, etc. (Harr, 1962; Strack, 1989, and before that by Polybarinova-Kochina, 1921). Römkens (2009) used this method in evaluating seepage and hydraulic potentials near incised ditches in shallow aquifers. More recently, he discussed in detail the work of van Deemter (1950) as it applies to drainage and infiltration into infinitely deep aquifers with tile lines (Römkens, 2013,and 2017) and incised ditches (Römkens, 2017). In practice, only in simple flow fields with straight line segments described as closed boundaries or curved open boundaries can this procedure be successfully used by utilizing special techniques such as the Schwarz-Christoffel transformation for a flow region where the boundaries are represented by straight line segments or by the Hodograph analysis for the case with open, curved boundaries. Both techniques were employed by van Deemter (1950). This article shows this technique, as used by van Deemter, for the rather simple flow field shown in Fig 1. Briefly, the flow region or aquifer is an infinitely deep strip with parallel vertical boundaries. These boundaries represent streamlines. He considered a drainage situation with a constant, uniform rainfall at the upper, open boundary and with a circular drain on one side that is fully filled with water. This drain ends in a submerged condition in a water conveyance system that is connected to an open water body. The adjoining tracts to this flow field are symmetric images of the tract under consideration. Fig.2 shows the sequence of conformal transformations used for this drainage case and the mathematical formulae used with $t=\sqrt{\sigma}/\lambda$. For details of these transformations, the reader is referred to the article by Römkens (2018) or the original work written in Dutch as a Government document by van Deemter (1950).While it is the objective to obtain z=f(ω), in many situations such a transformation is difficult if not impossible to obtain. In those cases it may be possible to use an intermediary approach in which z=f(t) and ω=g(t) in which t is the common parameter used for both transformations. Much about the analysis involving complex algebra can be learned from textbooks. In this regard, the books by Churchill (1960) and Strack (1989) can be very helpful. The conformal analysis as used by van Deemter (1950), yielded the following relationships for z and ω in terms of t:

$$z = a + i\cdot c + i\cdot\frac{a}{\pi}\cdot\left(\ln\frac{t-1-\beta}{t+1+\beta} + \frac{2}{\gamma}\cdot\ln\frac{t+1}{t+1+\beta} \right) \tag{1}$$

and

$$\omega = Kc + i\cdot Na + \frac{a}{\pi}\cdot[(L-N)\cdot(\ln(t-1) - L\cdot\ln(t-1-\beta) + N\cdot\ln(t+1+\beta) + (L-N)\cdot\frac{K-N}{K+N}\cdot\ln\frac{t+1}{t+1+\beta}] \tag{2}$$

where a, b, and c are the dimensions of the flow region or aquifer under consideration, N and L are the fluxes into (rainfall or seepage) or out of (evaporation or deep drainage) the flow region, and K is the hydraulic conductivity of the soil in the aquifer. Parameters γ and β are respectively defined as:

$$\gamma = \frac{(K+N)}{(L-N)} \quad \text{and} \quad \beta = -1 + \exp(\pi\gamma\cdot\frac{c-b}{2a}) \tag{3}$$

Physically, γ represents an intrinsic drainage characteristic reflecting the effect of rainfall (input) and transmittance (hydraulic conductivity) of the flow region. β represents the difference in groundwater level at the drain position and the point halfway or midpoint between two parallel drains reduced to unit drain spacing. The drain spacing is represented by 2a. The parameter β is also adjusted by the parameter γ. The origin of the flow region in the geometric plane(z = 0) is located at the vertex P and the corresponding value in the t-plane is given by t(P) = 1 while ω(P) ranges from –∞ +i·Na to –∞ + i·0. Then, for z = 0, relationship (1) becomes:

$0 = a + i\cdot c + i\cdot\frac{a}{\pi}\cdot(\ln\frac{-\beta}{2+\beta} + \frac{2}{\gamma}\cdot\ln\frac{2}{2+\beta})$

This relationship can be simplified after some algebraic manipulation to yield :

$$\pi\frac{c}{a} = \ln(1+\frac{2}{\beta}) + \frac{2}{\gamma}\cdot\ln(1+\frac{\beta}{2}) \quad (4)$$

Following the substitution of identity (3) for β in Eqn (4) one obtains:

$$\pi\frac{b}{a} = \ln(1+\frac{2}{\beta}) + \frac{2}{\gamma}\cdot\ln(\frac{1+½\beta}{1+\beta}) \quad (5)$$

Note the complex interdependence between b, c, and β.

DESCRIPTION GROUNDWATER TABLE

The location of the groundwater table is determined by a number of factors: (i) the water level in the drainage or conveyance system, (ii) the amount of water entering or leaving the aquifer at the soil surface(rain infiltration or evapo-transpiration) or the bottom of the flow region (seepage, drainage, or deep drainage), (iii) the hydraulic conductivity of the soil material in the aquifer, and (iv) the surface area of the drain or ditch through which groundwater exits or enters the water conveyance system or in other words the geometry of the flow field or of the aquifer. For a drainage flow regime as shown in Fig. 1 with adjoining mirror images of flow, the gradient of the groundwater table is zero above the drain (vertex Q) and halfway between two parallel drains (vertex R). To obtain the relationship for the groundwater table, represented by the open boundary QR in Fig. 1, the following expressions are introduced: (i) Express z in terms of polar coordinates, thus z = x +i·y = r·exp(i·(θ+2πk)), where k is an integer, $r = \sqrt{x^2+y^2}$, $\frac{x}{r} = \cos\theta$, and $\frac{y}{r} = \sin\theta$. (ii) Introduce the identity t = i·s , where s is a real number. Note that the groundwater table QR in the complex t-plane (Fig. 2) is represented by the imaginary axis in the complex t-plane and that the value of s varies from 0 at Q to ∞ at R. Then, the following relationship for z is obtained:

$$z = a + i\cdot c + i\cdot\frac{a}{\pi}[\ln\frac{i\cdot s-1-\beta}{i\cdot s+1+\beta} + \frac{2}{\gamma}\cdot\ln\frac{i\cdot s+1}{i\cdot s+1+\beta}] \quad (6)$$

The following identities were used to further simplify this relationship in order to obtain explicit expressions for x and y in terms of the parameters β, γ, and the variable s:

$$\ln(i\cdot s-1-\beta) = ½\ln[s^2+(1+\beta)^2] + \pi\cdot i - i\cdot \text{arctg}\frac{s}{1+\beta} \quad (7)$$

$$\ln(i\cdot s+1+\beta) = ½\ln[s^2+(1+\beta)^2] + i\cdot \text{arctg}\frac{s}{1+\beta} \quad (8)$$

$$\ln(i\cdot s+1) = ½\ln(s^2+1) + i\cdot \text{arctg}\, s \quad (9)$$

Substitution of the relationships (7), (8), and (9) into equation (6) yields:

$$z = x + i\cdot y = \frac{2a}{\pi}\{\frac{\gamma+1}{\gamma}\cdot \text{arctg}\frac{s}{1+\beta} - \frac{1}{\gamma}\cdot\text{arctg } s\} + i\cdot c + \frac{i\cdot a}{\pi\gamma}\ln\frac{s^2+1}{s^2+(1+\beta)^2} \quad (10)$$

This simplified relationship has a real and imaginary part which are associated with x and y, respectively. For a given value of s (a given point on the groundwater open boundary), the following x- and y-values are obtained:

$$\frac{x}{a} = \frac{2}{\pi}\cdot\{\frac{\gamma+1}{\gamma}\cdot \text{arctg}(\frac{s}{1+\beta}) - \frac{1}{\gamma}\cdot\text{arctg}(s)\} \quad (11)$$

$$\frac{y}{a} = \frac{c}{a} + \frac{1}{\pi\gamma}\cdot\ln(\frac{s^2+1}{s^2+(1+\beta)^2}) \quad (12)$$

Of specific interest are the vertices Q and R, which represent the locations of the groundwater table above the drain (Q) for which s = o and midway between the parallel drains (R) where s = ∞, respectively. The groundwater table at these locations is designated as b and c, respectively. Substitution of s = 0, representing the vertex Q in the t-plane, yields from Eqs. (11) and (12) the coordinates:

$$x = 0 \text{ and } y = c + \frac{a}{\pi\gamma}\cdot\ln\frac{1}{(1+\beta)^2}\text{, which, by definition, is b.} \quad (13)$$

Note, that the second term in the expression for y is a negative number which magnitude depends on the drain spacing and the value of β. The latter value is a quantity greater or equal to 0. Likewise, the vertex R in the t-plane (t =i· ∞ and s = ∞) yields the coordinates for R in the complex z-plane:

$$x = a \text{ and } y = c \quad (14)$$

The groundwater level c midway between the drain lines can also be determined from the requirement t(P) = 1 for which z = 0 or from Eqn 4.

GROUNDWATER REGIME

The groundwater regime in an open aquifer is at all times determined by the interplay between, on one hand, water entering the aquifer, represented by the parameters N and L (precipitation and sub-irrigation via tiles and ditches, seepage upward from deep groundwater sources, etc) and, on the other hand, water leaving the aquifer (evapo-transpiration , drainage, deep drainage). However, it also depends on the ability of the aquifer to hold and transmit water, or in other words on the hydraulic and intrinsic water conductive and storage properties of the soil, represented by the parameter K. The combined effect of these factors is represented by the parameter γ (Eqn 3).

Table 1.The effect (Δ) of β on the groundwater table above the drain and mid-way between drains.

Υ	β	c/a	b/a	Δ	γ	β	c/a	b/a	Δ
100	100	0.0313	0.0019	0.0294	100	75	0.0316	0.0040	0.0276

100	50	0.0326	0.0084	0.0242	100	25	0.0411	0.0203	0.0208
100	10	0.0694	0.0542	0.0152	100	1	0.3523	0.3478	0.0045
50	50	0.0540	0.0039	0.0501	50	25	0.0576	0.0576	0.0414
50	10	0.0808	0.0503	0.0305	50	1	0.3549	0.3460	0.0089
25	25	0.0908	0.0078	0.0830	25	10	0.1037	0.0426	0.0611
25	1	0.3692	0.3424	0.0168	10	10	0.1721	0.0194	0.1527
10	1	0.3755	0.3314	0.0441	5	1	0.4013	0.3131	0.0882

The complex inter-relationship between the parameters β and γ is clearly indicated upon examination of Eqn(3). While γ only depends on the rainfall intensity and the hydraulic conductivity of the soil in the aquifer, β depends both on the geometry of the wetted part of the aquifer (dimensions a, b, and c) and also on γ. Table. 1 summarizes some numerical values for c/a and b/a for several combinations of β and γ. The parameter Δ, defined as (c/a – b/a), represents the difference between the groundwater table mid-way between drain lines and the groundwater table above the drain line reduced to unit drain spacing. In utilizing Eqs.(4) and (5), one must be aware that the calculated b/a- and c/a-values may yield negative numbers. If so, then the acquired negative numbers suggest that the groundwater table is absent and that the combination of γ and β chosen are by definition inappropriate. The data indicate, that for large values of γ, the difference in the groundwater level between points above the drain and mid-way between the drain lines decreases with decreasing values of β. The groundwater level per se is higher at these points for lower β-values. Also, the groundwater levels are higher for aquifers with small γ-values. Water levels for which β = γ are impossible. As β → o, then the solutions for Eqs. (4) and (5) do not exist, because then b and c become ∞.

High values of γ denote, according to Eqn(3), when rewritten as $\gamma \approx K/N + 1$, a high value of the conductivity K or because of a small value of N (precipitation) or both. Then, there will be less water in the aquifer either because of rapid drainage or low precipitation. That means, that there is a larger difference between the ground water level above the drain (vertex Q) and the groundwater level above the mid-point (vertex R) between two parallel drains. In other words, β assumes a large value which can be attributed to both the large value for γ as well as a larger value for the term (b –c), representing the difference in groundwater level between the vertices Q and R. Conversely, low γ-values denote a low hydraulic conductivity or a high precipitation regime. Smaller γ-values also suggests a smaller value of β or a smaller difference between the groundwater level above the drain and that above of the mid-point between parallel drains.

EFFECT OF WATERLEVEL IN THE CONVEYANCE SYSTEM

Another factor that controls groundwater movement is the aquifer geometry itself such as the length of the flow pathways and the size and location of the water conveyance system in the aquifer. For the case at hand, in which excess water drains through a fully filled and submerged circular drain into a water conveyance system of ditches and streams, the drain water level also exercises a control on water release from the aquifer. In fact, the water pressure is determined by both the water level in the open water body, h_o, and the drain size and circumference through which drainage takes place. The tile drain is represented by the parameter r_o (the tile radius). Within the fully filled drain, the hydraulic potential is constant and consists of the water pressure, p_o ,at a point in the drain and the gravitational potential or location, h_o ,at that point relative to a known reference. Thus the total potential Φ and the potential function are then, respectively, defined as:

$$\Phi = \frac{po}{\rho g} + h_o \quad \text{and} \quad \varphi = K\Phi \tag{15}$$

An important point on the drain tile is $z(D) = -i \cdot r_o$ on the drain perimeter below vertex P. In the t-plane, this point is given b by $t = 1 + \delta < I + \beta$. The relationship between r_o and δ is given by:

$$-\frac{i \cdot ro}{a} = a + i \cdot c + i \cdot \frac{a}{\pi} \left[\ln\frac{\delta-\beta}{2+\delta+\beta} + \frac{2}{\gamma} \ln\frac{2 \div \delta}{2+\delta+\beta} \right] \tag{16}$$

Upon substitution the values for β and γ, one can determine δ . Also from Eqn(2) one can obtain an expression for φ_o :

$$\varphi_o = Kc + \frac{a}{\pi} \cdot [(L-N) \cdot \ln(\delta) - L \cdot \ln(\beta - \delta) + N \cdot \ln(2 + \beta + \delta) + (L - N) \cdot \frac{K-N}{K+N} \cdot \ln(\frac{2+\delta}{2+\beta+\delta})] \tag{17}$$

With the water level in the drain being $h_o = p_o/\rho g$ and the pressure in the drain being p_o then

$$\varphi_o = K \cdot (\frac{po}{\rho g} - r_o) = K \cdot (h_o - r_o)$$

Van Deemter used the above relationships to calculate δ and came up with the following table:

$\pi\frac{ro}{a}$	β	$\frac{ho}{a}$	δ
0.0128	9	—0.019	0.10
0.0128	4	0.031	0.028

Another method of determining the effect of water pressure in the conveyance-tile drain system on the groundwater table is the procedure described by Römkens (2009). In this procedure, the center of the drain (vertex P) which has a $-\infty$ pressure potential, is shifted along the imaginary axis of the flow field (towards vertex Q). Then, one can calculate the potential at the bottom of the drain with radius r_o designated as point $z(D_1) = - i \cdot r_o$ .Vertex D_1 should have the potential φ_o prevalent in the conveyance system. The flow to the drain in this asymmetrical flow field can then bere-calculated and must yield a certain value equal to Na. One now calculates the potential at the point $z(D_2) = i \cdot r_o$, which is the upper point of the drain. The potential at this point should be φ_o but found to be different due to the asymmetry of the flow field. By systematically changing the location of the vertex P along the imaginary axis y of the z-plane while maintaining the location and the potential at point D_1 one can arrive at potentials at point D_2 equal to φ_o . Since the value of the drain flow is a fixed quantity Na that does not change, the groundwater level in the aquifer must adjust. This iterative calculation procedure of shifting the location of the vertex P ultimately will lead to a situation of a constant value of the potential around the tile drain perimeter. A different value P(z) where the potential is –infinity, must lead to a flow field that reflects this change. In this manner the potential in the drain is maintained at the level commensurate with that of the real flow field. By using this procedure in this problem case, adjustments in the flow rate is contrary to the condition that q = Na. However the distribution could vary while maintaining the requirement of a total flow of Na. Changes in the flow distribution to the drain tile are then obtained through adjustments in the groundwater table in the aquifer. This particular approach needs further examination.

SUMMARY

The effect of aquifer characteristics on the groundwater table was examined for a steady rainfall regime, The analysis is based on the van Deemster (1950) analytical solution using conformal transformations. His approach offers significant potential to show how drainage flow may be affected by aquifer size and groundwater characteristics which in turn affects the groundwater table regime.

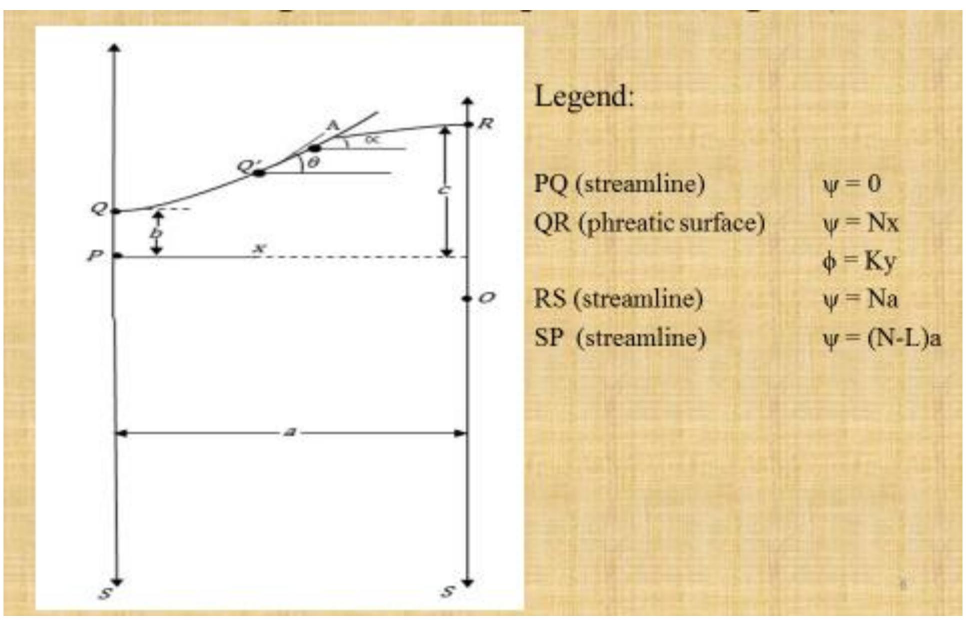

Fig. 1. A schematic representation of the flow region with Tile Drain.

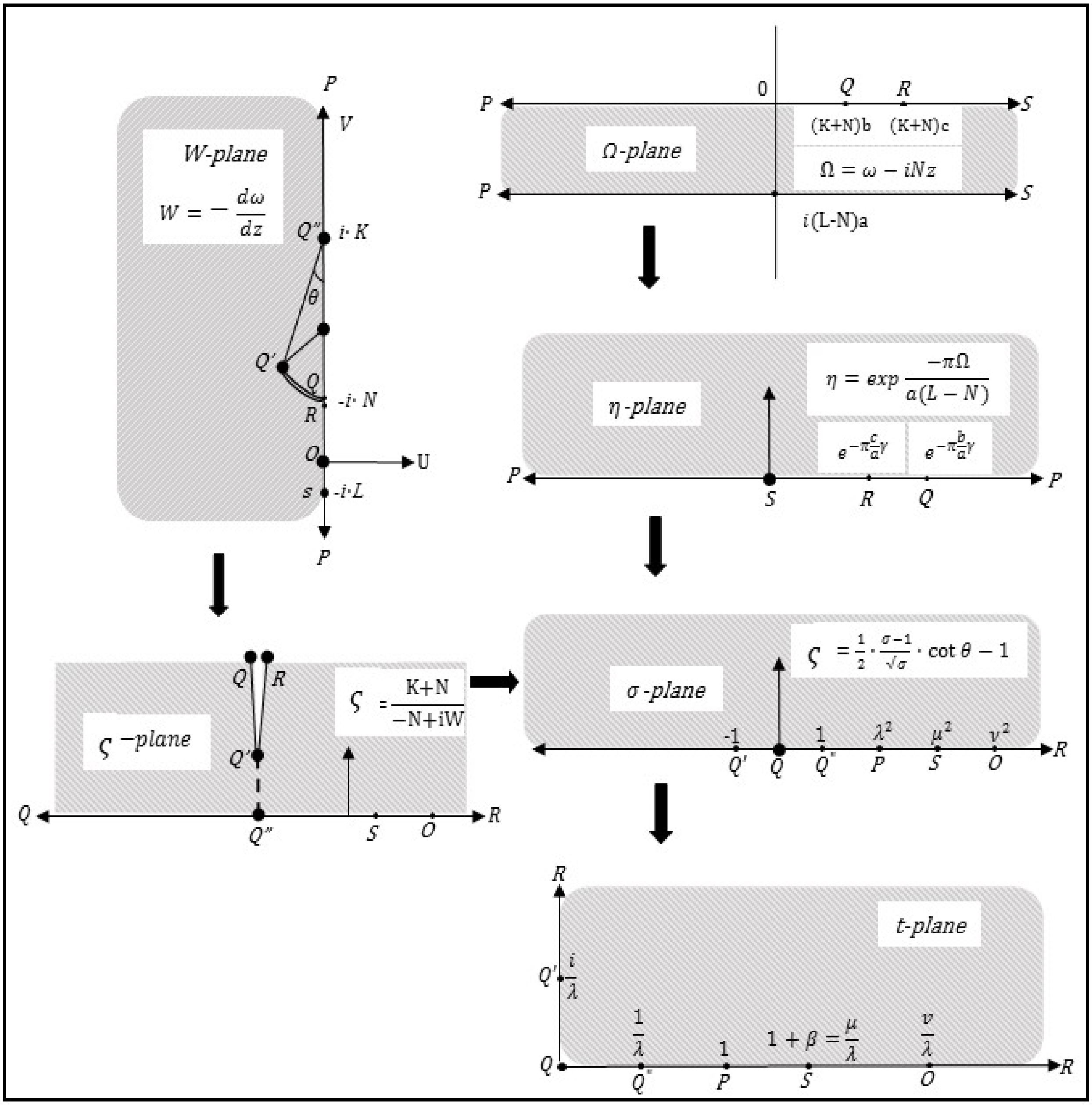

Fig.2. A graphical display of the transformations.

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